MATH 311: MATHEMATICAL MODELLING I
2024/2025

1. Objective (Minimize) $\Rightarrow Z = 3a + 5b$

Let "a" be the numbers of product A produces and "b" be the numbers of product B produces.

Constraints: $2a + 4b \leq 80$

$3a + 2b \leq 60$

$a + 3b \leq 36$

$a \geq 10$

$a + b \leq 40$, $a \geq 0$, $b \geq 0$

ⓐ $2a + 4b \leq 80$

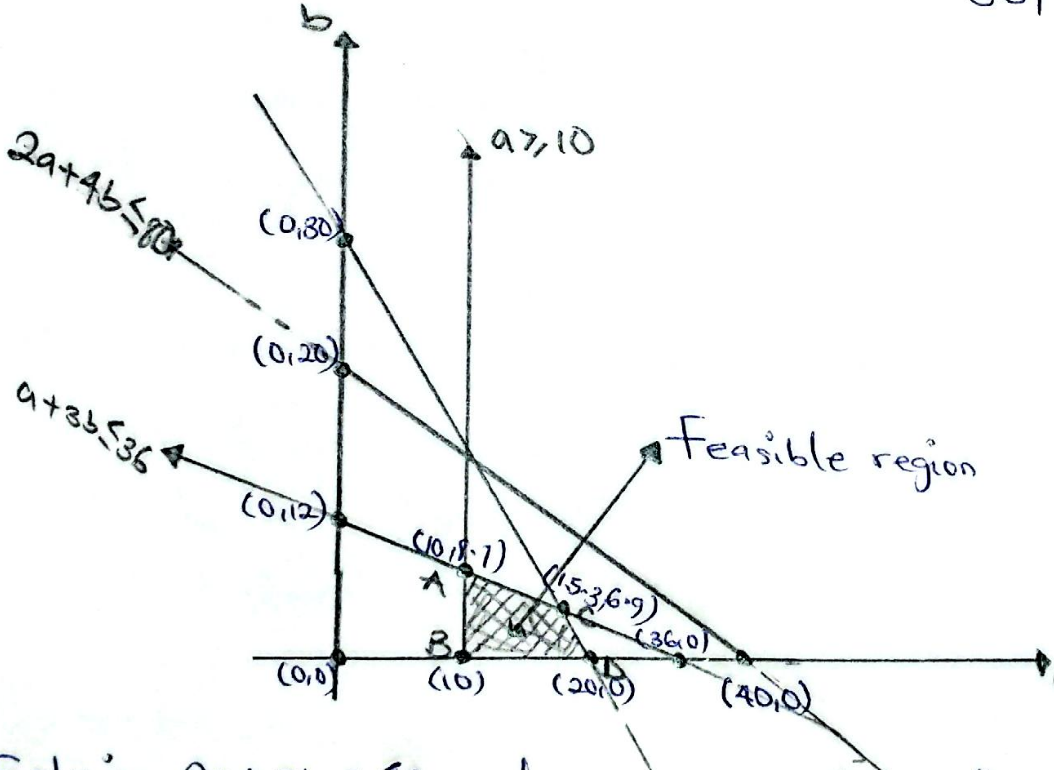
a	b
0	20
40	0

ⓑ $3a + 2b \leq 60$

a	b
0	30
20	0

ⓒ $a + 3b \leq 36$

a	b
0	12
36	0



Solving $3a + 2b \leq 60$ and $a + 3b \leq 36$ simultaneously,

we have $a = 15.3$ and $b = 6.9$ respectively.

~~Also, solving~~

Also, solving $a + 3b \leq 36$ and $a \geq 10$ simultaneously,

we have, $a = 10$ and $b = 8.7$ respectively.

Coordinates	a	b	$Z=3a+5b$
A(10, 8.7)	10	8.7	\$73.5
B(10,0)	10	0	\$30
C(15.3, 6.9)	15.3	6.9	\$80.4
D(20,0)	20	0	\$60

Target: Minimize total cost of production;

Optimization Value (cost) = \$30 and Optimization Soln ⇒ $a=10$ & $b=0$

Hence, I advise the factory owner to produce 10 products of A and 0 products of B in order to minimize cost. □.

2.) $y''-y'-6y=3x^2\exp(-2x)$ — ① Subject to $y(0)=1$ and $y(1)=5$

* Homogeneous Equation ⇒ $y''-y'-6y=0$ — ②

Let $y=e^{mx}$ ⇒ $y'=me^{mx}$ and $y''=m^2e^{mx}$ where $e^{mx}\neq 0$.

By substituting into eqn ②, we have;

$$m^2e^{mx}-me^{mx}-6e^{mx}=0$$

$$(m^2-m-6)e^{mx}=0 \quad \text{Since } e^{mx}\neq 0$$

$$\Rightarrow m^2-m-6=0 \Rightarrow (m+2)(m-3)=0 \Rightarrow m+2=0 \text{ or } m-3=0$$

$$\Rightarrow m=-2 \text{ or } m=3$$

Thus; $y_H=C_1e^{-2x}+C_2e^{3x}$.

* Particular Equation ⇒ $y_p=(Ax^2+Bx+C)xe^{-2x}$

$$=(Ax^3+Bx^2+Cx)e^{-2x}$$

$$y_p'=(3Ax^2+2Bx+C)e^{-2x}-2e^{-2x}(Ax^3+Bx^2+Cx)$$

and $y_p''=(6Ax+2B)e^{-2x}-2e^{-2x}(3Ax^2+2Bx+C)+4e^{-2x}(Ax^3+Bx^2+Cx)$

$$-2e^{-2x}(3Ax^2+2Bx+C)$$

By substituting y_p, y_p' and y_p'' into eqn ①, we have;

$$(6Ax+2B)e^{-2x}-2e^{-2x}(3Ax^2+2Bx+C)+4e^{-2x}(Ax^3+Bx^2+Cx)-2e^{-2x}(3Ax^2+2Bx+C)-e^{-2x}(3Ax^2+2Bx+C)+2e^{-2x}(Ax^3+Bx^2+Cx)-6e^{-2x}(Ax^3+Bx^2+Cx)=3x^2e^{-2x}$$

$6Ax - 12Ax^2 + \cancel{4Ax^3} + 2B - 8Bx + 4Bx^2 - 4C + 4Cx - 3Ax^2$
$+ \cancel{2Ax^3} - 2Bx + 2Bx^2 - C + 2Cx - \cancel{6Ax^3} - 6Bx^2 - 6Cx = 3x^2$

By Comparing bothsides, we have;

$A = -3/15$, $B = -3/25$ and $C = -6/125$

Thus; $y_p = -e^{-2x}\left(\frac{3}{15}x^3 + \frac{3}{25}x^2 + \frac{6}{125}x\right)$

Hence, the general solution of the equation becomes;

$y = y_H + y_p = C_1e^{-2x} + C_2e^{3x} - e^{-2x}\left(\frac{3}{15}x^3 + \frac{3}{25}x^2 + \frac{6}{125}x\right)$.

Using the conditions, $y(0) = 1$ and $y(1) = 5$, We then Substitute into the general solution to have;

at $y(0) = 1 \Rightarrow y = 1$ and $x = 0$

$C_1 + C_2 = 1$ —— (1)

and $y(1) = 5 \Rightarrow y = 5$ and $x = 1$

$\left(C_1 - \frac{46}{125}\right)e^{-2} + C_2e^3 = 5$ —— (2)

solving (1) and (2) simultaneously, we have;

$C_1 = 0.7537$ and $C_2 = 0.0263$

$\Rightarrow y = C_1e^{-2x} + C_2e^{3x} - e^{-2x}\left(\frac{3}{15}x^3 + \frac{3}{25}x^2 + \frac{6}{125}x\right)$

$= 0.7537e^{-2x} + 0.0263e^{3x} - e^{-2x}\left(\frac{3}{15}x^3 + \frac{3}{25}x^2 + \frac{6}{125}x\right)$

$= 0.0263e^{3x} - e^{-2x}\left(\frac{3}{15}x^3 + \frac{3}{25}x^2 + \frac{6}{125}x - 0.7537\right)$

3.a. An IVP (Initial Value problem) is an ODE together with an initial Condition which specifies the values of the Unknown function at a given point in the domain. Examples $y'' + 3y' - 2y = e^{2x}$ at $y(0) = 1$ and $y'(0) = 5$

A BVP (Boundary Value problem) is an ODE together with a boundary Condition which specifies the values of the Unknown function at a given point in the domain. Examples $y'' - y' - 6y = e^{2x}$ at $y(1) = 1$ and $y'(5) = 10$.

(3b). To find the laplace transform of;

i) t^3 Using $t^n = \frac{n!}{s^{n+1}}$ $\Rightarrow n = 3$

$$t^3 = \frac{3!}{s^{3+1}} = \frac{3!}{s^4} = \frac{6}{s^4}//$$

ii) $e^{(-7t)}$ Using $e^{-at} = \frac{1}{s+a}$ $\Rightarrow a = 7$

$$\Rightarrow e^{-7t} = \frac{1}{s+7}$$

OR

i) $t^3 = f(t) \Rightarrow L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty t^3 e^{-st}dt$ Using integration by Part, we have;

$u = t^3 \Rightarrow du = 3t^2$ and $dv = e^{-st} \Rightarrow v = -\frac{e^{-st}}{s}$

$$L[f(t)] = \cancel{-\frac{3t^2e^{-st}}{s}\Big|_0^\infty}^{\,0} + \frac{1}{s}\int_0^\infty 3t^2e^{-st}dt = \frac{3}{s}\int_0^\infty t^2e^{-st}dt$$ Repeating the steps, we have;

$u = t^2 \Rightarrow du = 2t$ and $dv = e^{-st} \Rightarrow v = -\frac{e^{-st}}{s}$

$$L[f(t)] = \frac{3}{s}\left[\cancel{-\frac{t^2}{s}e^{-st}\Big|_0^\infty}^{\,0} + \frac{2}{s}\int_0^\infty e^{-st}.t\,dt\right] = \frac{6}{s^2}\int_0^\infty te^{-st}dt.$$

$u = t \Rightarrow du = dt$ and $dv = e^{-st} \Rightarrow v = -\frac{e^{-st}}{s}$

$$L[f(t)] = \frac{6}{s^2}\left[\cancel{-\frac{te^{-st}}{s}\Big|_0^\infty}^{\,0} + \frac{1}{s}\int_0^\infty e^{-st}dt\right] = \frac{6}{s^3}\int_0^\infty e^{-st}dt = \frac{6}{s^3}\left[-\frac{e^{-st}}{s}\Big|_0^\infty\right]$$

$$L[F(t)] = \frac{6}{s^4}\left[-e^{-st}\Big|_0^\infty\right] = \frac{6}{s^4}\left[\cancel{-e^{-s(\infty)}}^{\,0} + e^{-s(0)}\right] = \frac{6}{s^4}[1]$$

Hence,

$$L[f(t)] = L[t^3] = \frac{6}{s^4}//$$

ii) $F(t) = e^{-7t} \Rightarrow L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-7t}.e^{-st}dt = \int_0^\infty e^{-(7+s)t}dt$

$$= -\frac{e^{-(7+s)t}}{7+s}\Big|_0^\infty = -\frac{1}{s+7}\left[e^{-(7+s)t}\Big|_0^\infty\right]$$

$$= \frac{-1}{s+7}\left[e^{-(7+s)\infty} - e^{-(7+s)(0)}\right] = \frac{-1}{s+7}(-1) = \frac{1}{s+7}$$

Hence,

$$L[f(t)] = L[e^{-7t}] = \frac{1}{s+7}//$$

④ $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial y^2}$; $t \leq 0$: $U = 0$ for $y \geqslant 0$; $t > 0$: $\begin{cases} u = 1 \text{ at } y = 0 \\ u = 0 \text{ as } y \to \infty \end{cases}$

$L\left(\frac{\partial u}{\partial t}\right) = \alpha L\left(\frac{\partial^2 u}{\partial y^2}\right) \Rightarrow L\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty e^{-st} \frac{\partial u}{\partial t} = e^{-st} u - \int u \cdot -s e^{-st}$

$= ue^{-st}\Big|_0^\infty + s\int_0^\infty ue^{-st}$

$= -u + s\bar{u}$ — ⊛

$\alpha L\left(\frac{\partial^2 u}{\partial y^2}\right) = \alpha \int_0^\infty e^{-st} \frac{\partial^2 u}{\partial y^2} dt = \alpha \frac{\partial^2}{\partial y^2} \int_0^\infty u e^{-st} dt$

$= \alpha \frac{\partial^2}{\partial y^2} \bar{u}$ ------ ⊛

Substiting in the question;

$s\bar{u} - u = \alpha \frac{\partial^2 \bar{u}}{\partial y^2} \Rightarrow \alpha \frac{\partial^2 u}{\partial y^2} - s\bar{u} = 0$

$\Rightarrow \alpha m^2 - s = 0 \Rightarrow m^2 = s/\alpha \Rightarrow m = \pm\sqrt{s/\alpha}$

$\Rightarrow \bar{u} = Ae^{\sqrt{s/\alpha}\, y} + Be^{-\sqrt{s/\alpha}\, y}$ at $u = 1$, at $y = 0$

$\Rightarrow L(1) = \int_0^\infty e^{-st} dt = \int_0^\infty e^{-st} = -\frac{1}{s} e^{-st}\Big|_0^\infty = +1/s$ //

$\Rightarrow \frac{1}{s} = Ae^0 + Be^0 \Rightarrow \frac{1}{s} = A + B$ — ①

$\bar{U} = 0 \Rightarrow Ae^{\sqrt{s/\alpha}\,\infty} + Be^{-\sqrt{s/\alpha}\,\infty}$

$\Rightarrow Ae^\infty + 0 = 0 \quad \cdot A = 0, \quad B = 1/s$

$\bar{u}(y s) = \frac{1}{s} e^{-\sqrt{s/\alpha}\, y}$

$u(yt) = L^{-1}\left(\frac{1}{s} e^{-\sqrt{s/\alpha}\, y}\right) = \operatorname{erfc}\left(\frac{y}{2\sqrt{\alpha t}}\right)$

(5) To find the second degree polynomial $y = ax^2 + bx + c$ Using the least squares method

x	y	$X = x-2.5$	X^2	X^3	X^4	Xy	X^2y
1	1.1	−1.5	2.25	−3.38	5.06	−1.65	2.48
1.5	1.3	−1	1	−1	1	−1.3	1.3
2	1.6	−0.5	0.02	−0.13	0.06	−0.8	0.03
2.5	2.0	0	0	0	0	0	0
3	2.7	0.5	0.25	0.13	0.06	1.35	0.68
3.5	3.4	1	1	1	1	3.4	3.4
4	4.1	1.5	2.25	3.38	5.06	6.15	9.23
	16.2	0	6.77	0	12.24	7.15	17.12

Substituting into the following formular below;

$\Sigma y = a\Sigma X^2 + b\Sigma X + \Sigma c$, $\Sigma Xy = a\Sigma X^3 + b\Sigma X^2 + c\Sigma X$.

$\Sigma X^2y = a\Sigma X^4 + b\Sigma X^3 + c\Sigma X^2$ we have

$16.2 = 6.77a + 7c$ —— ①　　　$7.15 = 6.77b \Rightarrow b = \frac{7.15}{6.77} = \frac{715}{677}$

$17.12 = 12.24a + 6.77c$ —— ②　　　$= 1.0561$

Solving ① and ② simultaneously, we have;

$a = \frac{145}{569} = 0.2548$ and $c = 2.0679$

Hence, the second degree polynomial $y = ax^2 + bx + c$

$$y = 0.2548X^2 + 1.0561X + 2.0679$$

or $$y = 0.2548(x-2.5)^2 + 1.0561(x-2.5) + 2.0679.$$

6a) Using the method of Undetermined Coefficients to solving

$$\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 24y = e^{-3x} \quad \text{——} ①$$

Homogeneous Equations $\Rightarrow \frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 24y = 0$ —— ②

If $y = e^{mx} \neq 0 \Rightarrow y' = me^{mx}$ and $y'' = m^2e^{mx}$　　$\boxed{y' = \frac{dy}{dx} \text{ and } y'' = \frac{d^2y}{dx^2}}$

By substituting into eqn ②, we have;

$m^2e^{mx} + 11me^{mx} + 24e^{mx} = 0$

$(m^2 + 11m + 24)e^{mx} = 0$　if $e^{mx} \neq 0$ then $m^2 + 11m + 24 = 0$

$(m+3)(m+8)=0 \Rightarrow m+3=0$ or $m+8=0 \Rightarrow m=-3$ or $m=-8$

$y_H = C_1 e^{-3x} + C_2 e^{-8x}$.

The partcular equation $\Rightarrow y_p = Axe^{-3x}$

$\Rightarrow y_p' = Ae^{-3x} - 3Axe^{-3x}$ and $y_p'' = -3Ae^{-3x} - 3Ae^{-3x} + 9Axe^{-3x}$

By substituting into eqn ①, we have;

$-6Ae^{-3x} + \cancel{9Axe^{-3x}} + 11Ae^{-3x} - \cancel{33Axe^{-3x}} + \cancel{24Axe^{-3x}} = e^{-3x}$

$5Ae^{-3x} = e^{-3x} \Rightarrow 5A = 1 \Rightarrow A = 1/5$

Thus, $y_p = \frac{x e^{-3x}}{5}$

Hence,

$y = y_H + y_p = C_1 e^{-3x} + C_2 e^{-8x} + \frac{x}{5} e^{-3x}$

$= C_2 e^{-8x} + C_3 e^{-3x}$ where $C_3 = C_1 + \frac{x}{5}$.

(6b) Recall, $y = Ce^{-kt}$

$y_1 = \$320{,}000$, $t_1 = 0$ }—— initial Condition

$\Rightarrow C = 320{,}000$

$y_2 = \$286{,}000$ $t_2 = 2005 - 2010 = 5$

$\Rightarrow 286000 = 320{,}000\, e^{-5k} \Rightarrow k = \frac{-1}{5}\ln\left(\frac{143}{160}\right) = +0.0225$

$t = ?$ $y = \$0$ (near)

Since exponential functions never exactly reach 0, let's define liquidation as when revenue drops below \$1000.

$\Rightarrow y = 320{,}000 e^{-0.0225t}$

$\Rightarrow 1000 = 320000 e^{-0.0225t}$

$\Rightarrow t \approx 256.37$

$\Rightarrow 256.4 = t$

By Converting t to year, we have;

year $= 2005 + 256.4 = 2261.4$

$\simeq 2261$

Hence, the company will be liquidated around the year 2261

Department of Mathematics
Ahmadu Bello University, Zaria

MATH 311-Mathematical Modelling I, Time: 2hrs. 2024/2025 Session

Answer FOUR Questions, Question 1 is COMPULSORY

1. A factory which produces two products A and B intends to minimize the total cost of production while satisfying multiple constraints. The raw materials required to produce A and B are 2kg and 4kg respectively while each unit of A and B respectively cost \$3 and \$5 to produce. Also, each unit of A and B require 3hrs and 2hrs of labour respectively. The factory has a total of 80kg raw material and 60hrs labour. In addition, the company has a total of 36 machine hours available where A requires 1 machine hour and B, 3 machine hours. If at least 10 unit of A must be produced and the total units to be produced must not exceed 40 due to limited storage, advise the factory on the number of each product to produce.

2. Use the method of undetermined coefficients to find the general solution to $y'' - y' - 6y = 3x^2 \exp(-2x)$ subject to $y(0) = 1$ and $y(1) = 5$.

3a. Explain the differences between an IVP and BVP with examples.

b. Find the Laplace transform of the following functions:

i. t^3 ii. $exp(-7t)$.

4. Use the Laplace transform technique to obtain the solution of

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial y^2}; t \leq 0 : u = 0 \text{ for } y \geq 0; t > 0 : \begin{cases} u = 1 \; at \; y = 0 \\ u = 0 \; as \; y \to \infty \end{cases}$$

5. Fit a second degree polynomial to the following data using the least squares method. Find the coordinate of the minimum or maximum point.

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

6a. Use the method of undetermined coefficients to find the solution to the ODE;

$$\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 24y = e^{-3x}.$$

b. A slow economy caused a company's annual revenue to drop from \$320,000 in 2005 to \$286,000 in 2010. If the revenue follows an exponential pattern of decline, when will the company be liquidated?